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Nonlinear dynamics of uniformly loaded *Elastica*: Experimental and numerical evidence of motion around curled stable equilibrium configurations

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It has been numerically observed and mathematically proven that for a clamped Euler's *Elastica*, which is uniformly loaded, there exist, in large deformations, some 'undocumented' equilibrium configurations which resemble a curled pending wire. Even if *Elastica* is one of the most studied model in mathematical physics, we could not find in the literature any description of an equilibrium like the one whose existence was forecast theoretically in [36].

In this paper, we prove that this kind of equilibrium configurations can be actually observed experimentally when using 'soft' beams. We mean with soft beams: *Elasticae* whose ratio between the applied load intensity and the bending stiffness is large enough. Moreover, we prove experimentally that such equilibrium configurations are actually stable, by observing their oscillations around the considered nonstandard equilibrium configuration.

To describe theoretically such oscillations we consider, instead of a 'soft' *Elastica* model, directly a Hencky-type discrete model, *i.e.* a 'masses-springs' finite dimensional Lagrangian model. In this way we formulate, avoiding the use of an intermediate continuum model, a model for which numerical simulations can be performed without the introduction of any further discretization. In this way, we can also predict quantitatively the motions of soft beams, in the regime of large displacements and deformations. Postponing to future investigations more careful quantitative measurements, we report here that it was possible to get a rather promising qualitative agreement between observed motions and predictive numerical simulations.

KEYWORDS
discrete modelling, hencky bar-chain, nonlinear beam

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"Frustra fit per plura quod fieri potest per pauciora."
(It is pointless to do with more what can be done with fewer.)
In Summa Totius Logicae *by William of Ockham*

1 | INTRODUCTION

The fundamental works by Bernoulli and Euler,[25,63] introduced the concept of *Elastica* in mathematical physics. This model has been the basis of many mathematical investigations and leads to important technological applications. It constitutes from the mathematical point an interesting source of theoretical and experimental problems[27] while it had applications to structural mechanics which had a tremendous impact. The literature in which the theory of Euler–Bernoulli beam theory and/or its applications are studied is so huge that it is probably impossible to give a reasonable account for it. Among the many relevant problems which have been studied with great details and care we, simply as they are instrumental to our investigations, cite [8,13,62,71,94,102,105].

However, the importance of the model is so great that there are still left interesting and practically significant problems to be studied anew or whose study needs to be completed. In particular, the problem of large deformations of an Euler beam under loads distributed along its length seems not to have attracted the attention which it deserved. This problem played a relevant role in the theory of metamaterials and in particular when dealing with pantographic microstructures with long beams.[12,29,34,35,40,48,51,61,104] In pantographic sheets[6,43,69] the single fiber, in a certain class of situations, can be modelled as a beam interacting with the other beams/fibers, in such a way that in the homogenized limit, a distributed load can be assumed to be applied. Actually in pantographic structures with long 'fibres' (see, *e.g.*, Figure 1), we can observe the existence of 'long and soft beams' undergoing large deformations and subjected to 'distributed' loads. Of course, such beams are not always exactly subjected to uniformly distributed load! However, it was believed that a better knowledge of the possible equilibrium shapes for these load cases was necessary to proceed in the investigations about pantographic metamaterials. It has to be remarked that, when studying large deformations of uniformly loaded Euler beams one is led to non-autonomous variational problems, *i.e.*, to variational problems whose corresponding Euler-Lagrange equations are non-autonomous differential equations. These problems cannot be solved easily, standing the available mathematical techniques: in fact in [36] it could be shown that: i) there exist non-trivial equilibrium shapes having a curled shape for clamped Euler beams under uniformly distributed dead load orthogonal to the initial straight undeformed configuration ii) many interesting properties of the global and local minimizers of the total energy for Euler's *elastica* hold, in the case of large displacements and deformations equilibria.

Recently a reassessment of the models needed to describe beam-like structural elements has been used in order to simplify the tools needed to study nonlinear equilibrium shapes, loss of stability, buckling and post-buckling phenomena like those studied in [64,83,84,95]. In particular in order to simplify the computation of the buckling load of a beam, its large deformations and its multiple stable equilibria discrete Hencky models have been recovered.[16,17,111,114] Indeed Hencky, in his seminal work [72],

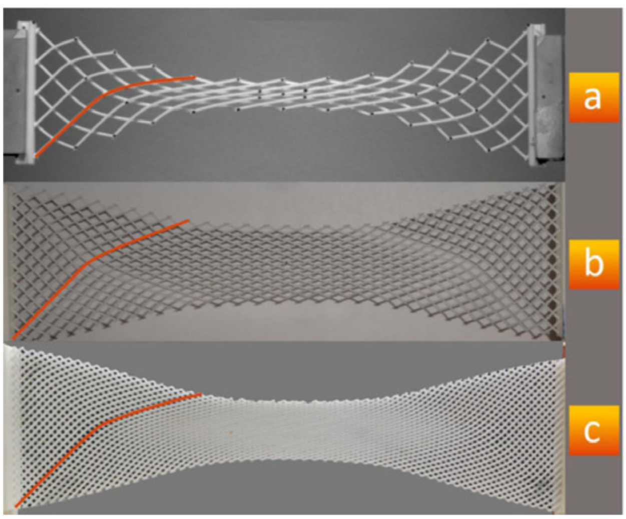

FIGURE 1 Some illustrative examples of pantographic structures under a bias test varying the density of the fibers

for making possible some semi-analytical solutions of the problem of finding large deformations of Euler beams, proposed a very clever and effective way leading to the computation of the buckling load of a rectilinear planar beam.[1]

The idea, in a nutshell, is based on the choice of modeling a beam as a system of rigid linking elements interconnected by elastic joints. Using Lagrangian generalized coordinates and Lagrangian potential energy for the elastic joints by minimizing total energy for the so obtained system one manages to find a finite set of equilibrium equations, *i.e.* the conditions imposing that the gradient of the total potential energy is vanishing. In this way, one gets rather simply an effective estimate of the buckling load. Obviously, such an estimate has an increasing accuracy when one increases the number of elastic joints and rigid links.

From a conceptual point of view, possibly the most attractive feature of Hencky's procedure consists in the fact that it avoids at all the necessity of introducing continuum models. As continuous models, to get numerical results in large deformations, need, at the end, to be discretized one gets a relevant economy in the modeling procedure. Indeed, the Hencky model is discrete since the very beginning and it has a clear mechanical interpretation. When applied to systems which need generalized continua, to use discrete models may also avoid some sharp debates.[39,41,45,47,49,50]

To make this discussion complete, and make happy those who consider continuum models more important, we can recall here that recently some Γ-convergence results have been proven that Hencky's model is a fully reliable approximation of continuous inextensible[5] and extensible[4] Euler–Bernoulli beams. Actually generalized continua can be proven to be Γ-limits of systems described at micro-level as first gradient continua.[2,3,91,92] On the other hand, it is possible to regard a beam as an energetic boundary curve on a two-dimensional manifold in a three-dimensional space and employ the frameworks developed in [73,77,78] and even account for higher gradients elaborated in [75]. The advantage of this approach is that the fictitious bulk material can regularize the behavior of the beam and this allows to analyze buckling in a computational framework. Furthermore, the substrate provides a versatile medium to assign distributed loads on the beam similar to thin films on elastic foundations.[74,76,89]

A further remark is needed here: the introduced discrete model is intrinsically nonlinear. It naturally incorporates geometrical and material non-linearities: this is the reason for which, also when one linearizes it, he can avoid all the issues implied by the loss of objectivity of the energy when, in the neighborhood of a certain equilibrium configuration, the linearization is performed.

Limiting our attention to mechanical problem and choosing, in the literature those works whose spirit is closer to ours, the application of Hencky's modeling concept can be found again in some more recent works which deal with single beams, see [10,11,26,32,58,81,99,100,106], or with assemblages of beams, see [44,48,59,69,107,112]. Moreover Hencky-type models can be used to conceive specially targeted lattice materials structured at different scales, see [22]. An additional reason suggesting to consider Hencky's discrete models is given exactly by the consideration of the aforementioned architectured multiscale materials (whose microstructure is often modeled as an assembly of beams, [28,54,55,101]). Indeed, this kind of micro-structured materials are constituted by a very large number of structural elements, and therefore some reduced order masses-springs models can be useful for their description.

It has to be also remarked that when one studies complex metamaterials,[21,29,39,44,48,53,86,87] and if one wants to use generalized continuum models finally he is led to the problem of synthesis: *i.e.* the problem of determining a microstructure governed, in a suitable homogenization limit, by postulated equations. This synthesis is finally obtained by means of lattices structures, which can be regarded as a kind of generalization of the discrete systems introduced by Hencky. As these discrete models have been proven to be really efficient in predicting the mechanical behavior of considered structures, (see, *e.g.*, [57] where different shapes of the unit cells are considered, [79,80] for triangular lattices and [56,96] for chiral lattices) one can really question the need of talking at all about continuum models. We can, therefore, deduce that the understanding of the pattern of deformations of Euler beams, which are the fundamental structural elements considered in meta-material micro-structures, is a very important topic in the theory of metamaterials, may one decide to model them as *e.g. Elasticae* or as Hencky-type discrete systems.

These general considerations apply also to the more particular class of pantographic (micro)structures,[44,109] particularly useful when looking for materials showing an elastic behavior in the range of large displacements and large deformations, see [107,110]. It has to be remarked that the effectiveness of Hencky-type models allows for the inclusion, in the modelling scheme, also of the models for the onset of any form of damage phenomenon eventually leading to failure phenomena.[103,108]

In the literature, one can find some examples of works dealing with the nonlinear dynamics of beams, modelled as discrete models (see for instance [59,68,81,112]). Therefore, the present work is aimed to contribute to an effort which has been already initiated. One of the main novelty consists in having adapted the approach already presented in the literature to the aim of describing very simple (but probably not already considered) physical experiments, which can be reproduced without expensive tools: we mean the construction of soft (highly flexible) beams, their placing in a curled equilibrium shape under a gravity load and the investigation of their planar nonlinear motion in the neighborhood of this 'nonstandard' configuration.

[1] For completeness we recall here that Hencky's idea is already present in the works of Gabrio Piola almost one century before, see [37,46], together with the main ideas needed to get via homogenization, from discrete Lagrangian systems, generalized, and eventually non-local, continua (see [15,38]).

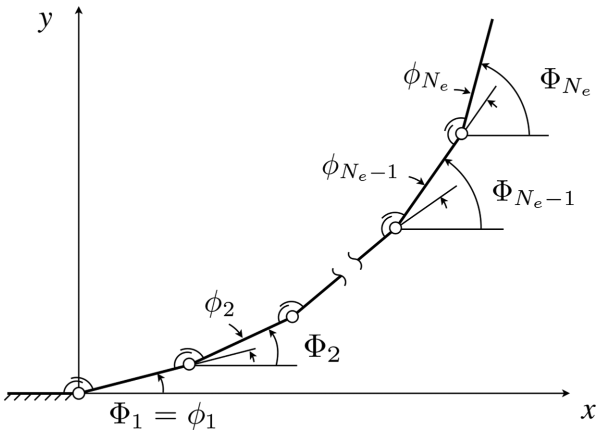

FIGURE 2 Hencky-type lumped mass-spring model for a highly flexible beam

We assume, in formulating our Hencky-type model, the same hypotheses which are peculiar of the continuous Euler beam model: i) the conservative part of deformation energy (*i.e.* elastic energy) depends upon the relative Hencky-bars rotations (*i.e.* curvature); ii) the Hencky bars are rigid (*i.e.* the axis of the Euler beam is inextensible); iii) because of the simplicity of the Hencky model then the concept of shear deformation cannot even be formulated when considering the discrete model, this situation corresponds to the assumption that the the cross-section with respect to the axis of the used soft beams is negligible; and that this cross-section must be regarded to be not deformable.

As we will see with numerical experiments, Hencky's model is consistent with the assumptions usually accepted for formulating Euler beam model.

For determining the equilibrium shapes of soft beams, and the numerical simulations we can best fit mechanical parameters necessary to specify the particular model for the specific soft beam used in the experimental setup. Using these parameters, *i.e.* the masses and the stiffnesses of rotational elastic joints, and fitting subsequently only the damping coefficients we can describe, by running efficient and robust numerical codes, rather effectively the observed mechanical behavior and can highlight the main and peculiar features of the dynamic nonlinear behavior of the considered class of soft beams.

The plan of the paper is the following: after this short introduction, the main technical ideas which necessary to construct the used discrete Hencky model are described in Section 2. In the subsequent Section 3, we carefully describe the experimental set-up which we have used and the two specific experiments which were performed. The obtained data are then fully reported, making thus possible the detailed comparison between experimental evidence and the predictions given by the numerical simulations which were performed, both for static equilibrium configurations and for the observed motions. Finally, in Section 4 the paper is concluded, by discussing the impact of our main results in the static and dynamics of metamaterials together with the possible extensions of this work to suitably newly designed linear metamaterials.

2 | A HENCKY-TYPE MODEL REPLACING THE *ELASTICA* MODEL

We want to describe the static and dynamic behavior of a 'soft' cantilever beam to which a gravity load is applied, in the most simple possible way, having in mind the need of describing a specific set of experimental pieces of evidence (see next section).

The choice made here, assuming a point of view accepted also by Piola,[37,38,41,46] is based exactly on a similar choice made by Hencky. One could consider Hencky-type discrete model as an independent model, having its own physical meaning and its own numerical advantages, or as a discrete approximation of Euler, *e.g.*, *Elastica*.

By simply adapting the Hencky technique[98,111,114] we choose to consider a discrete system made of an 'articulated' sequence of N_e (rigid, as the beam is assumed inextensible) rods of length η, which are constrained at their terminal points by perfect hinges, *i.e.* hinges which are exerting vanishing couples. To each joint we assume that a rotational spring is applied, aiming to model the elasticity effects which are resistant to the bending of the considered mechanical system. (see Figure 2). Each Lagrangian configuration of the just described system is fully specified once it is known the time evolution of the N_e Lagrangian angular coordinates, $\Phi_i(t)$. These coordinates determine the orientation of each of the considered rigid rods relative to the x-axis which is chosen to be horizontal (the applied loads will be assumed to be vertical). The direction along which the applied loads are assumed to be directed coincides with the y-axis, which is oriented vertically upwards.

The set of bars, before the deformation, is assumed to be straight along the x-axis. The springs will have their rest length such that this configuration will have the minimum of deformation energy.

To each rigid segment, it is attributed a mass, m_i, together with a moment of inertia, J_i, relatively to a rotation axis which is orthogonal to the plane in which the motion is assumed to be constrained. This rotation axis is assumed to pass through the center of mass of the bar.

The position of the mass center for each segment is given by the following formula:

$$\begin{cases} x_i(t) = \sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \cos(\Phi_k(t)) \\ y_i(t) = \sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \sin(\Phi_k(t)) \end{cases} \tag{1}$$

where δ_{ik} is the Kronecker delta.

In order to evaluate the velocities of said mass centers, one must differentiate with respect to time, getting the following formulae:

$$\begin{cases} \dot{x}_i(t) = -\sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k(t) \sin(\Phi_k(t)) \\ \dot{y}_i(t) = \sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k(t) \cos(\Phi_k(t)) \end{cases} \tag{2}$$

The adopted Lagrangian parameters are suitable to produce a really convenient expression for the equations of motions which we will derive starting from the Action Principle applied using as a starting point the following Lagrangian, which, also because of the analysis by Hamilton, and specifically by Piola (see [37]) for Euler beams, we believe is suitable in the considered physical instance

$$\mathscr{L} = \mathfrak{K} - \Psi \tag{3}$$

where $\mathfrak{K}$ and Ψ are the kinetic and potential energies of the system, respectively, as determined by the theory of constrained rigid body motions with elastic joints (see, *e.g.*, [14,70]).

It is easy to accept that the potential energy Ψ can be obtained by two terms: an elastic Ψ_{el} and a gravitational Ψ_{wg} term. The former elastic term Ψ_{el} is postulated as follows

$$\Psi_{el} = \sum_{i=1}^{N_e} \kappa_{b\,i} \left[\cosh(\phi_i) - 1\right] \tag{4}$$

where the relative angles between adjacent rods, ϕ_i, were introduced. Because of the clamping constraint we have: $\phi_1 = \Phi_1$ while $\phi_i = \Phi_i - \Phi_{i-1}$ for $i \geq 2$. Moreover, we have assumed that the concentrated bending stiffnesses $\kappa_{b\,i}$, associated to the rotational springs[43,107] are all equal (uniformity of bending stiffness condition) and that the applied external loads reduce to the gravitational term Ψ_{wg} given by

$$\Psi_{wg} = \sum_{i=1}^{N_e} g\, m_i \left[\sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \sin(\Phi_k)\right] \tag{5}$$

where g is the acceleration of gravity.

In order to be self-contained, we recall the the kinetic energy, using König's theorem, can be proven to have the following algebraic expression

$$\mathfrak{K} = \sum_{i=1}^{N_e} \frac{1}{2} m_i \left\{ \left[\sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \sin(\Phi_k)\, \dot{\Phi}_k\right]^2 + \left[\sum_{k=1}^{i} \eta \left(1 - \frac{\delta_{ik}}{2}\right) \cos(\Phi_k)\, \dot{\Phi}_k\right]^2 \right\} + \frac{1}{2} J_i \dot{\Phi}_i^2 \tag{6}$$

Because of physical plausibility, the only restriction on the potential energy in Equation (4) requires that it must be positive definite: therefore any convex function which could be necessary can be postulated. This means that we can expect that some specific meta-beams may be designed being characterized by energies which are not quadratic functions of the relative angles. The choice in Equation (4) has been made for consistency reasons with the corresponding continuum model, but it is remarkable that the first non-vanishing term in its Taylor expansion valid for small relative variations of angles is exactly the standard usual quadratic form.

We will need to take into account also a possible viscous dissipation, as in our experiments the observed oscillations are damped. (we refer *e.g.* to [9,33,66,82] for an interesting discussion of the concepts needed here). This dissipation occurs mainly, during the motion, because of the interaction of the 'soft' beam with the surrounding air. Therefore we postulate, as a first conjecture and therefore neglecting internal dissipations, a Rayleigh dissipation function of the following form:

$$\mathfrak{R} = \sum_{i=1}^{N_e} \frac{1}{2} c_{b\,i}\, \dot{\phi}_i^2 \tag{7}$$

In conclusions The Euler-Lagrange stationarity condition for chosen action produces the following Lagrange equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{K}}{\partial \dot{\Phi}_i}\right) - \frac{\partial \mathfrak{K}}{\partial \Phi_i} + \frac{\partial \Psi_{el}}{\partial \Phi_i} + \frac{\partial \Psi_{wg}}{\partial \Phi_i} + \frac{\partial \mathfrak{R}}{\partial \dot{\Phi}_i} = 0 \quad \text{for } i = 1 \ldots N_e \tag{8}$$

The Equations (8) do not present any particular difficulty. It is indeed a system of ordinary differential equations whose existence and uniqueness theorems are well established as are easily available efficient and robust numerical codes. Completely differently would be the nature of the problem to be solved if we had decided to describe the kinematics of our system with an infinite dimensional space of configurations. In this case, the Euler-Lagrange equations governing the motion would have been a partial differential equation whose difficulty is much greater.

We solve numerically the found discrete Lagrange equations in the framework of the computing system Wolfram Mathematica. The solver we have chosen is the one called differential-algebraic system of equations (DAEs). To use it we needed to apply a proper transformation to get an equivalent normal form, $F_j(t, \Phi_i, \dot{\Phi}_i) = 0$. This transformation produces a (larger) set of differential equations which is, instead of the second order, of the first order in the time variable.

3 | EXPERIMENTAL SET-UP, EXPERIMENTAL EVIDENCE AND OBTAINED NUMERICAL SIMULATIONS

The previously introduced Lagrangian model is used, in the present section to obtain predictions to be tested with two sets of experimental measurements. To be more precise, we have used two different soft beams constituted by two different materials and having different sizes and lengths. These two soft beams have been sized in order to be assured that the nonstandard equilibrium 'curled' configurations were forecast to be really local minima for their total energy. Moreover, we have chosen the properties of the beams in such a way that the dominant damping factor was actually the friction interaction with the air during the imposed oscillations. In this way, it was possible to investigate successfully the dynamic behavior of used soft beams also in the regime of large oscillation around their effectively observed curled stable equilibrium configurations.

The first used soft beam was constituted by a paper strip. Its dimensions were 329×20 mm, and its thickness was about 0.17 mm. Finally, its mass has been evaluated to be 0.62 ± 0.02 g.

The second used soft beam was constituted by a rather similar strip but made by a thin isotropic sheet of polyethylene terephthalate, *i.e.* standard PET. Its dimensions were 220×20 mm and thickness was about 0.15 mm. The mass of the strip was 0.91 ± 0.02 g. To show up further effects (which can be easily predicted by simply adding a term in the Lagrangian) we have loaded the free end of the considered beams with another 'concentrated' mass. We simply attached a 'metallic paper clip' of mass $m_{pc} = 0.41 \pm 0.02$ g and moment of inertia $J_{pc} = (7.5 \pm 0.5) \times 10^{-9}$ kg m^2 passing through a barycentric axis and orthogonal to the plane of motion (see Figure 3). Specifically, to take into account this paper clip, we add to the Lagrangian function, the gravitational potential energy

$$\Psi_{pc} = g\, m_{pc} \sum_{k=1}^{N_e} \eta \sin(\Phi_k) \tag{9}$$

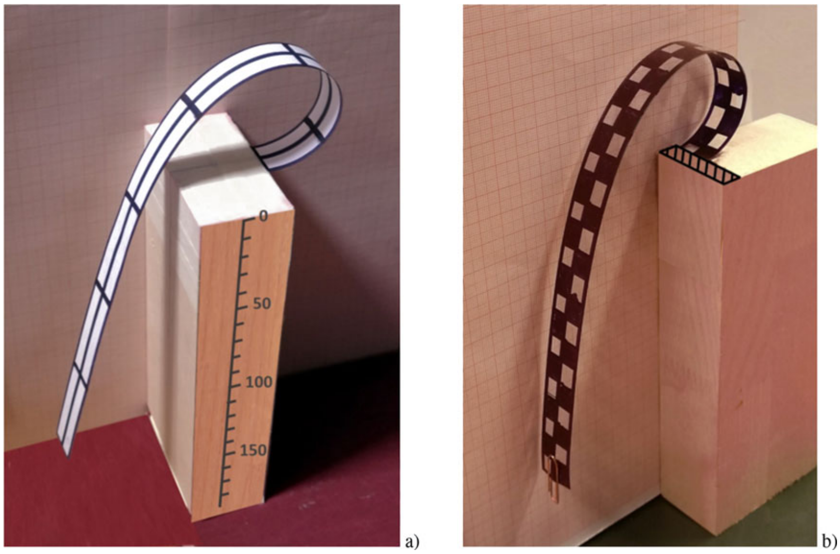

FIGURE 3 Stable nonstandard 'curled' equilibrium shapes: paper beam **a**; PET beam with a mass on the tip **b**

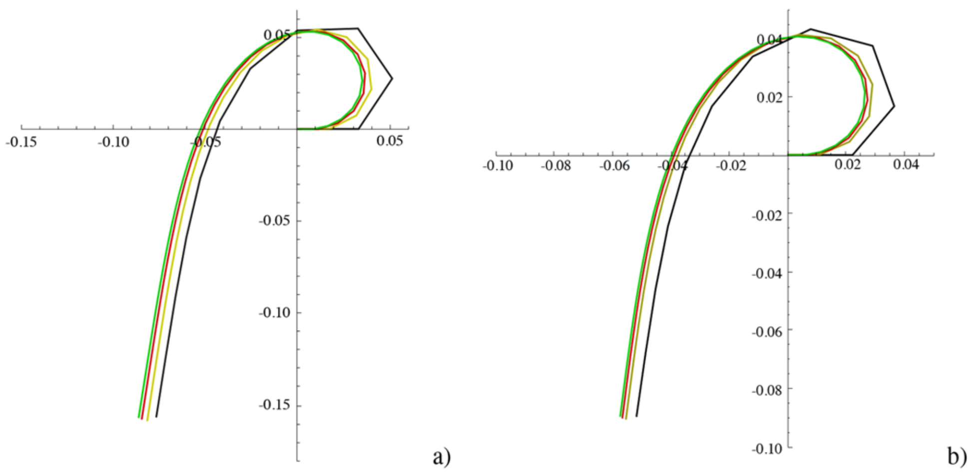

FIGURE 4 Convergence analysis of curled stable equilibrium configurations obtained by numerical simulations: paper beam **a**; PET beam with a mass on the tip **b**. The curled stable equilibrium configurations represented here are discretized in $N_e = \{10, 20, 30, 40\}$ elements

and coherently the kinetic energy

$$\mathfrak{K}_{pc} = \frac{1}{2} m_{pc} \left\{ \left[\sum_{k=1}^{N_e} \eta \sin(\Phi_k)\, \dot{\Phi}_k \right]^2 + \left[\sum_{k=1}^{N_e} \eta \cos(\Phi_k)\, \dot{\Phi}_k \right]^2 \right\} + \frac{1}{2} J_{pc} \dot{\Phi}_{N_e}^2. \tag{10}$$

To make the visualization of the shapes easier in the figures we have painted on one elastic strip some lines, while we have colored alternatively with rectangles the other strip.

3.1 | Fitting of the elastic parameters by comparison with the experimental measures of the curled static configurations

The homogenization results available in the literature assure that the Hencky-type model converges to the Euler–Bernoulli continuous model.[5] In our cases, as $N_e = 30$ and $\eta = 11.0$ mm for the paper strip while $\eta = 7.3$ mm for the PET strip, the discrete approximation is close enough to the continuous model. Figure 4 shows a clear convergence analysis made by changing the number of rigid discrete elements from 10 to 40. It is pretty evident that solutions with 30 and 40 elements are very close, thus we chose $N_e = 30$ to avoid unnecessary computational burden.

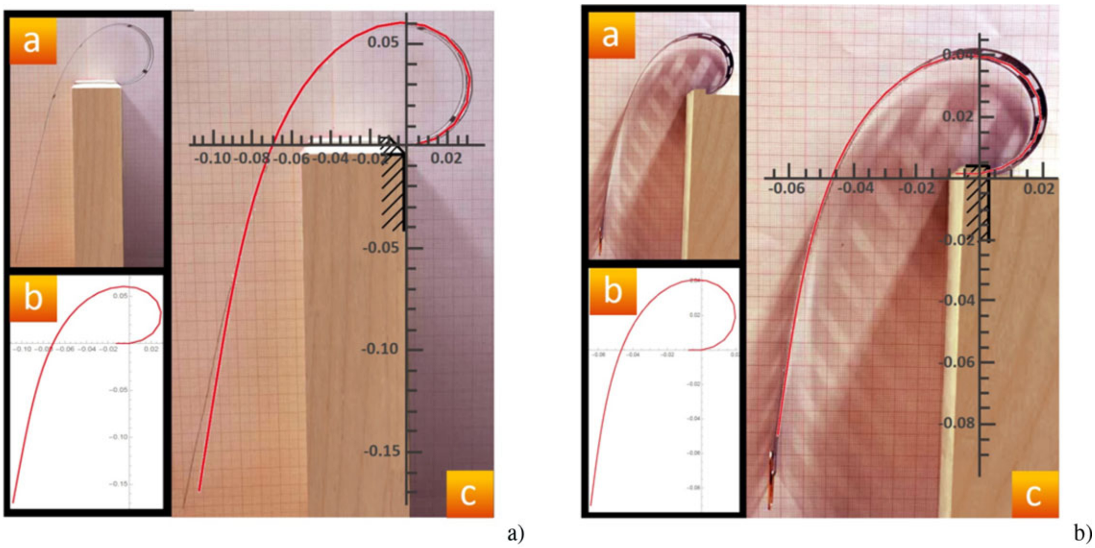

FIGURE 5 Experimentally observed and numerically evaluated equilibrium configurations: paper beam **a**; PET strip with a tip mass **b**

On the other hand, the available solutions to the De Saint Venant problem and the results of homogenization by [24] assure that the Euler bending stiffness can be approximated with the Young modulus of the material constituting the beam and the moment of inertia of its section. Also, this approximation seems rather precise for our comparative purposes. Therefore, we will estimate the bending stiffness coefficient for the rotational spring in the Hencky model by assuming that in this spring there is the same deformation energy which one can find in the corresponding beam element, as calculated with the De Saint Venant solutions. However, we are aware of the fact that the *a priori* estimate of the Young modulus of used paper and used PET strips could be too approximate to be useful. Instead much more precise can be the *a priori* estimate of the moment of inertia of considered strips. Therefore, we decided to get the Young moduli by means of the best fit of the equilibrium curled shapes as measured in our static experiments (see Figure 5).

Indeed, to start with, we considered the local-minimum energy configurations which we called ‘curled’ for the two soft beams used in our experimentation (see Figure 3). In the second series of figures we have overlapped to the image of the true physical object the calculated curve of equilibrium for the Hencky-type model (recall it is a Polygonal chain constituted by 30 segments). To be more precise: they show the photographic pictures of the curled equilibrium shapes of the physical stripes used in the experiment overlapped to the theoretical equilibrium configurations found by calculating the second lowest local minimum of the total energy Ψ as evaluated with $N_e = 30$.

The agreement exhibited in Figure 5, could be finally obtained by assuming for the lumped bending stiffness an expression which is a trivial consequence of the previously listed assumptions:

$$Y_b = \kappa_{b\,i}\eta\,(j_b)^{-1} \tag{11}$$

where j_b is the second moment of area of the beam cross section. The obtained Y_b, as estimated by imposing the overlapping of the figures by varying in a ‘reasonable’ range the values for $\kappa_{b\,i}$, can be regarded as the best approximation of the the Young modulus of the considered materials, which we can obtain.

This value will be used now on, in this paper, as it identifies the most correct material parameter. Particularly, we found for the paper beam the best fitted value is given by $Y_b = 1.25$ GPa while for the PET beam such value is given by $Y_b = 2.5$ GPa. Of course we started, in the guessing procedure, from tentative values in an interval including to the otherwise measured values of the Young moduli for paper and PET.

We remark here that the constitutive law (4) is just one of the possible choice to describe the behavior of the considered soft beam. As a matter of fact, many other constitutive laws can be adopted as, for instance, a quadratic function of the ‘deformation’ angle ϕ_i with the expression:

$$\Psi_{el}{}^{Quad} = \sum_{i=1}^{N_e} \kappa_{b\,i}\,\frac{\phi_i{}^2}{2} \tag{12}$$

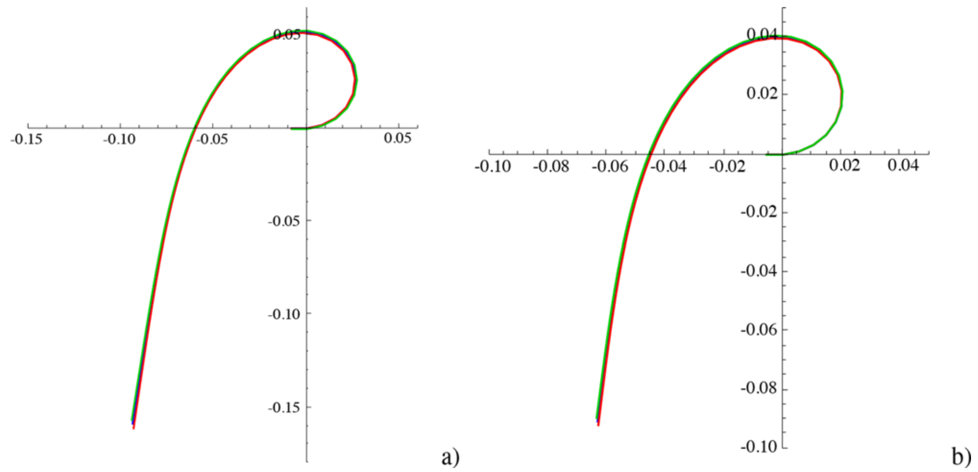

FIGURE 6 Curled stable equilibrium configurations with different constitutive laws: quadratic elastic potential (blue solid line); cosine elastic potential (red solid line); hyperbolic cosine elastic potential (green solid line). Paper beam **a**; PET beam with a tip mass **b**. The curled stable equilibrium configurations represented here are discretized in $N_e = 40$ elements

TABLE 1 Comparison between Hencky and Euler-Bernoulli model for the paper cantilever strip: natural frequencies

	mode I	mode II	mode III	mode IV
Hencky model (Hz)	0.3687	2.3121	6.4774	12.7002
Euler-Bernoulli model (Hz)	0.3810	2.3877	6.6857	13.1014
Relative error (%)	–3.23	–3.17	–3.12	–3.06

or also a trigonometric dependence on the same angle as follows:

$$\Psi_{el}{}^{Trg} = \sum_{i=1}^{N_e} \kappa_{b\,i} \left[1 - \cos(\phi_i)\right] \tag{13}$$

Indeed, the law (12) is the most commonly adopted one, while the law (13) some times is preferable because it is easier for computing purpose when Cartesian coordinates are used as Lagrangian variables but in the validity range between $-\pi/2$ and $\pi/2$, values excluded.The choice of the proper constitutive law is quite difficult and deserves a careful procedure based on experimental observation. For instance, procedures which make use of the χ-square test can be employed to select the proper law. It is also worthy to say that if the deformations are small, all the considered laws produce the same results since they have the same main leading term in a Taylor expansion. In our case, we compared the curled stable equilibrium configurations obtained for the three mentioned laws (see Figure 6) and thus, we can establish that even if the displacements and the rotations are large, the deformations, on the contrary, are almost in a small range. For this reason, in the present study we can adopt any law among those considered.

3.2 | Free linearized vibrations of the undamped soft beams

For the sake of dynamic analysis, and to check the convergence of the discrete formulation proposed, we also perform a modal linearized analysis of the two examined cases. A first investigation has been performed comparing the results of the proposed Hencky-type formulation with those obtained by the standard Euler–Bernoulli model. By neglecting the gravitational potential and assuming a discretization of $N_e = 30$ rigid elements, we can easily compare the natural frequencies of the cantilever beams for the two above-mentioned formulations in the undamped case. Tables 1 and 2 summarize these results, limited to the first four modes, for the paper strip and for the PET beam with a point mass attached to its tip, respectively. As it can be seen in these tables, the relative error between the continuous and the discrete model, with only 30 elements, at most is about 3 %, therefore we have a satisfactory accuracy even with relatively few elements.

TABLE 2 Comparison between Hencky and Euler-Bernoulli model for the PET cantilever strip: natural frequencies

	mode I	mode II	mode III	mode IV
Hencky model (Hz)	0.3953	3.2433	9.8290	19.9172
Euler-Bernoulli model (Hz)	0.4002	3.2644	9.9438	20.3689
Relative error (%)	–1.22	–0.65	–1.15	–2.22

TABLE 3 Natural frequencies of free vibrations around the global and local equilibrium shape in the paper strip case

	mode I	mode II	mode III	mode IV
Global equilibrium (Hz)	1.147	3.225	6.466	11.949
Local equilibrium (Hz)	1.250	1.629	4.141	8.701

TABLE 4 Natural frequencies of free vibrations around the global and local equilibrium shape in the PET strip case

	mode I	mode II	mode III	mode IV
Global equilibrium (Hz)	1.285	4.298	9.217	18.463
Local equilibrium (Hz)	1.455	1.944	6.003	13.509

Subsequently, the free vibrations of the considered soft beams have been analyzed around equilibrium configurations. In particular, in the selected cases we found that each beam under a uniform gravitational load presents two stable equilibrium configurations which are solutions of the following equation:

$$\left.\frac{\partial \Psi}{\partial \Phi_i}\right|_{\bar{\Phi}} = 0 \qquad \forall\, i = 1, ..N_e \tag{14}$$

The two stable solutions ($\bar{\Phi}$) of Equation (14) are a global minimum and a local one: the former being the ‘classical’ solution of a cantilever beam which resembles a hanging wire, and the latter is the ‘curled’ shape shown in Figure 3. To study the small vibrations around these equilibrium shapes, we linearize the equations of motion or in other words, we reduce the potential energy Ψ and kinetic energy $\mathfrak{K}$ to quadratic forms considering the leading terms (of order two) in a Taylor expansion of these energies as computed below

$$\Psi^* = \frac{1}{2} \left.\frac{\partial^2 \Psi}{\partial \Phi_i \partial \Phi_j}\right|_{\bar{\Phi}} \Phi_i \Phi_j \tag{15}$$

$$\mathfrak{K}^* = \frac{1}{2} \left.\frac{\partial^2 \mathfrak{K}}{\partial \dot{\Phi}_i \partial \dot{\Phi}_j}\right|_{\bar{\Phi}} \dot{\Phi}_i \dot{\Phi}_j \tag{16}$$

From the matrices associated to the quadratic forms (16) and (15), *i.e.*,

$$M_{hk} = \left.\frac{\partial^2 \mathfrak{K}}{\partial \dot{\Phi}_h \partial \dot{\Phi}_k}\right|_{\bar{\Phi}} \qquad \text{and} \qquad U_{hk} = \left.\frac{\partial^2 \Psi}{\partial \Phi_h \partial \Phi_k}\right|_{\bar{\Phi}}, \tag{17}$$

we may compute the eigenvalues and the eigenvectors associated and report them in the Tables 3 and 4 and in Figures 7 and 8, respectively, for both cases: the paper and the PET strip with a point mass on the tip. Specifically, we note that comparing the first natural frequencies of the global minimum and of the local minimum, the frequency of the local minimum solution is larger than the other one. Thus, in a sense, it is as if the vibrating length for the curled solution is smaller than the other one and for this reason, the beam behaves a bit more stiffly. We observe also that the natural modes, in Figures 7 and 8, exhibit a number of nodes, which are resting points, progressively growing with the number of modes as it is common in vibrating problems.

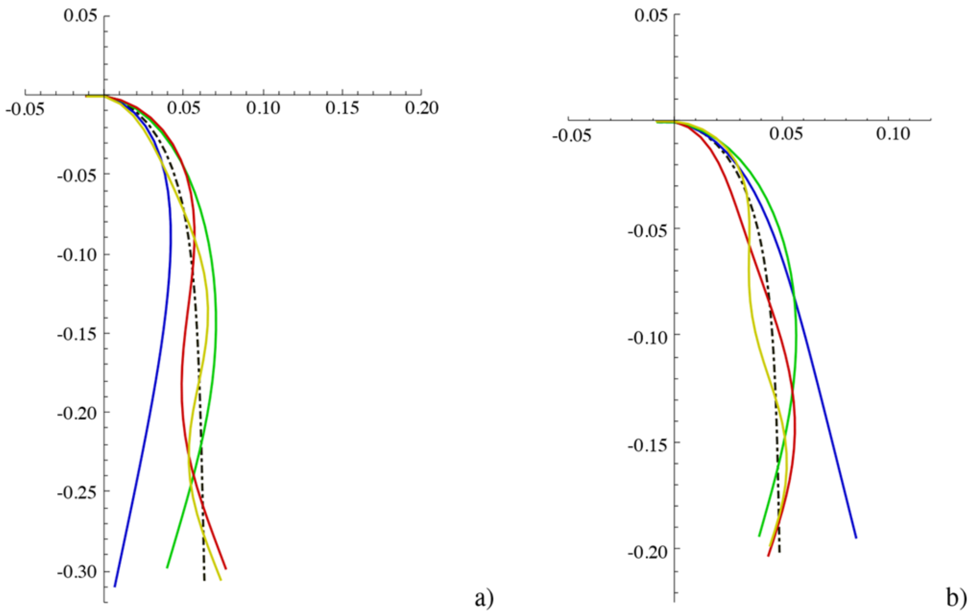

FIGURE 7 First four natural modes (solid lines) around the global equilibrium shape (dot-dashed line). Paper beam **a**; PET beam with a mass on the tip **b**. Legend: blue line corresponds to the first mode; green line to the second mode; red line to the third mode; yellow line to the fourth mode

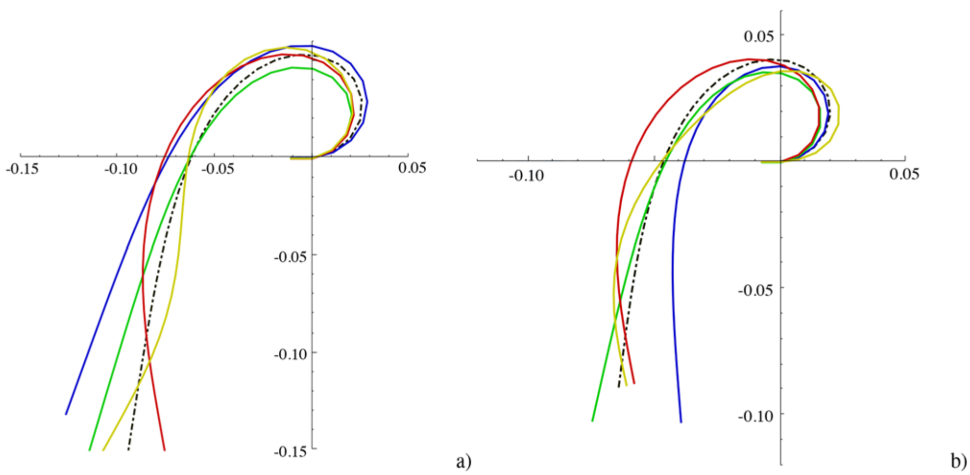

FIGURE 8 First four natural modes (solid lines) around the local 'curled' equilibrium shape (dot-dashed line). Paper beam **a**; PET beam with a mass on the tip **b**. Legend: blue line corresponds to the first mode; green line to the second mode; red line to the third mode; yellow line to the fourth mode

3.3 | Motions in the neighborhood of the curled stable equilibrium configurations: large oscillations

To prove experimentally that the observed equilibrium curled configuration is indeed stable it is necessary to observe the small motions starting in its neighborhood and establish that they are not increasing definitively the actual configuration from the curled equilibrium. No observed small perturbation has caused such a definitive removal. The used numerical code is capable to solve any initial value problem by forecasting the evolution of any in-plane oscillating motion consequent to any initial displacement of the extreme tip of the strip. To make simple the experimental determination of the needed initial data, we have considered only motions with initial zero velocity.

From the experimental pictures of said initial configuration, we evaluate the equilibrium configuration corresponding to the imposed placement of the terminal strip tip. By minimizing the energy and by using the already determined constitutive parameters we can calculate the effective Lagrangian coordinates of the observed initial configurations.

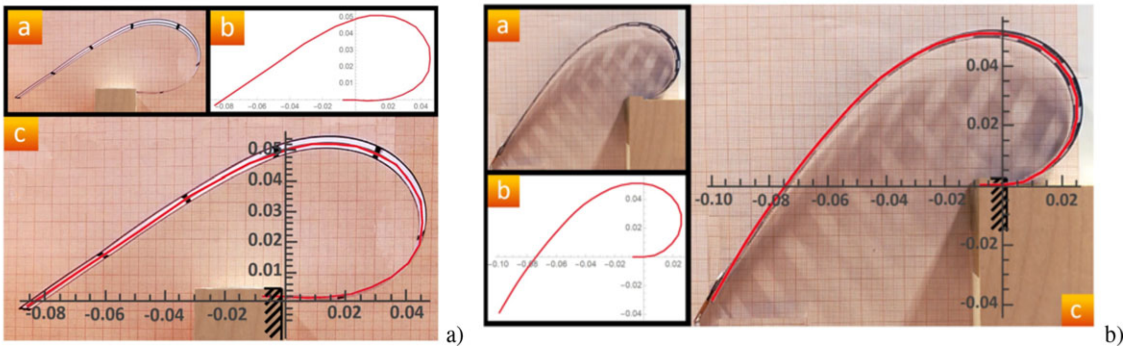

FIGURE 9 Experimentally imposed and numerically evaluated initial configurations: paper beam **a**; PET beam with a mass on the tip **b**

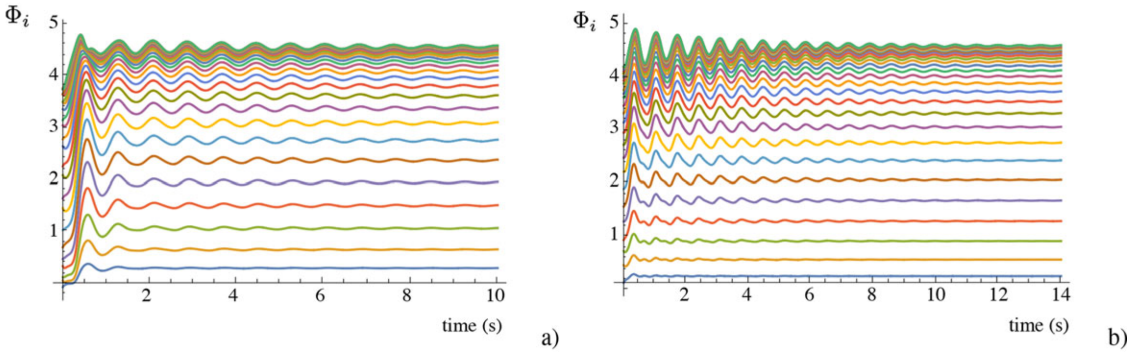

FIGURE 10 Time history of the Lagrangian coordinates, $\Phi_i(t)$, around the curled equilibrium configurations for the paper case **a**, and for the PET case **b**. Their plots result in an increasing order from the $\Phi_1(t)$ to $\Phi_{Ne}(t)$

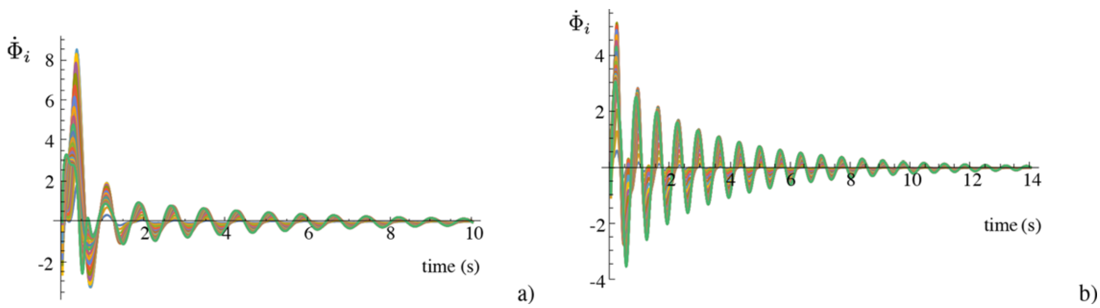

FIGURE 11 Time history of the angular velocities, $\dot{\Phi}_i(t)$ for the paper case **a**, and for the PET case **b**

In Figure 9 we can observe the effective overlap of the computed piecewise linear curve predicting the shape of the strip and the real picture of the both the strips considered. One has to remark that by using always and only the few parameters determined with the first fitting, the one which allowed for the prediction of the curled shape, we have obtained always a very close correspondence between experimental measurements and theoretical predictions. In the second step of the procedure we have specified as initial values these calculated equilibria, together with a condition of vanishing of all initial Lagrangian velocities.

By solving the ODEs given by Equation (8) we can predict the motions which originate from the experimentally imposed initial conditions. In order to present the obtained computed motions, in terms of the time evolutions of the orientations $\Phi_i(t)$ and their derivatives velocities $\dot{\Phi}_i(t)$ we have elaborated the Figure 10 and Figure 11, respectively. They account for both the two cases studied here: that is the paper and the PET strip beams.

Figure 11 is then presented to show, by reducing the time interval, the initial part of the predicted motion.

There are some remarks which are made possible by the results of our numerical simulations. These remarks may be considered as some interesting conjectures which may conduct to some rigorous demonstrations.

1. The considered initial shapes and the calculated equilibrium configurations, towards which the system during its evolution, is approaching, are all curves whose curvature has a constant sign. This means that said curves are not crossing their tangents. This has as a consequence that the coordinates $\Phi_i(t)$, in every instant t, are not decreasing with the increasing index i. Indeed, we remark that Figure 10 shows the calculated values of considered Lagrangian coordinates. The corresponding plots show how the sequence $\Phi_1(t)$ to $\Phi_{Ne}(t)$ is not decreasing.
2. Always Figure 10 shows that when the index i is increasing, the values of $\Phi_i(t)$ tend to be constant. This means, geometrically, that the final part of the beam tends to remain straight.
3. In Figure 11 we see the plots of Lagrangian (in this case angular) velocities. These velocities have a maximum value in the beam segments characterized by low values of the index i: *i.e.* when the considered segments of the vibrating beams are close to the clamped one and when their characterizing abscissa s is smaller than $L/4$. The velocities are larger at the beginning of the calculated motions. Subsequently, the dissipation phenomena become dominant and the time evolution produces all vanishing Lagrangian velocities. This means that all plots of Lagrangian velocities are increasingly superimposed and crowded. A simple inspection of Figure 11 allows us to establish which segment has the highest velocity at every time instant.

All presented numerical simulations incorporate a description of the dissipation phenomena which are occurring during the performed experiments. Said simulations were performed by using some estimated viscosity coefficients for the considered soft beams obtained. This estimate was obtained by means of a first, rather rough, but surprisingly efficient, criterion. We observed that the stopping time of the vibrating beams can be reasonably well detected by using the videos which were recorded during the experiments, even if there is, at the moment, a large sensitivity error. A simple best-fit determination of viscosity parameters has been performed to get a numerical estimation of the stopping times which are in a reasonable agreement with the measured times.

In particular we have found that

$$c_n = 2.0 \times 10^{-5}\ \mathrm{N\,m\,s}$$

for the strip constituted by paper and

$$c_n = 7.0 \times 10^{-5}\ \mathrm{N\,m\,s}$$

when the strip constituted by PET is considered.

Once this first estimate has been successfully obtained it was possible to produce the following Figure 12. In it, one can find the plots describing the evolution, during the motion, of the total kinetic energy and the total potential energy for both the paper and the PET strips. To compare these calculated quantities with some experimentally measured physical quantity will require the development of more sophisticated experimental set-ups. However one can analyze them qualitatively. Their time evolution is exactly showing the expected features: i) they evolve in counter-phase, *i.e.* in time there is an oscillation of the beams in which kinetic energy is transformed into potential energy and vice versa ii) the kinetic energy decreases and finally tends to zero, as clearly had to happen because of dissipation iii) the potential energy approaches its value, as calculated for the local minimum which is estimated for the equilibrium configuration to which the beam is approaching after the needed number of damped oscillations.

To resume the whole calculated time history of Lagrangian coordinates we use the Figure 13. This figure is conceived to show the trajectory in the so-called phase space of the Lagrangian coordinate of the terminal Hencky segment of considered beams, *i.e.* for both the paper strip and the PET strip. In other words, we present the phase space trajectory of the Lagrangian coordinate $\Phi_{Ne}(t)$ for both the investigated cases. We do not find meaningful to show the similar trajectories relative to the other Lagrangian coordinates as they are very similar. We remark that: i) in the initial part of both the trajectories both motions are strongly affected by nonlinear phenomena ii) mentioned nonlinearities are more relevant for the paper strip when compared with the PET strip iii) towards the final part of the oscillations both systems become substantially linear and one can easily detect the configuration of stable equilibrium.

4 | SOME CONCLUDING REMARKS AND FUTURE CHALLENGES WE EXPECT TO CONFRONT

The motivations from which this paper had its origins are to be found in the study of fundamental constituents of the microstructure of pantographic metamaterials. It was surprising to discover that in the literature: i) problems of motion and deformation

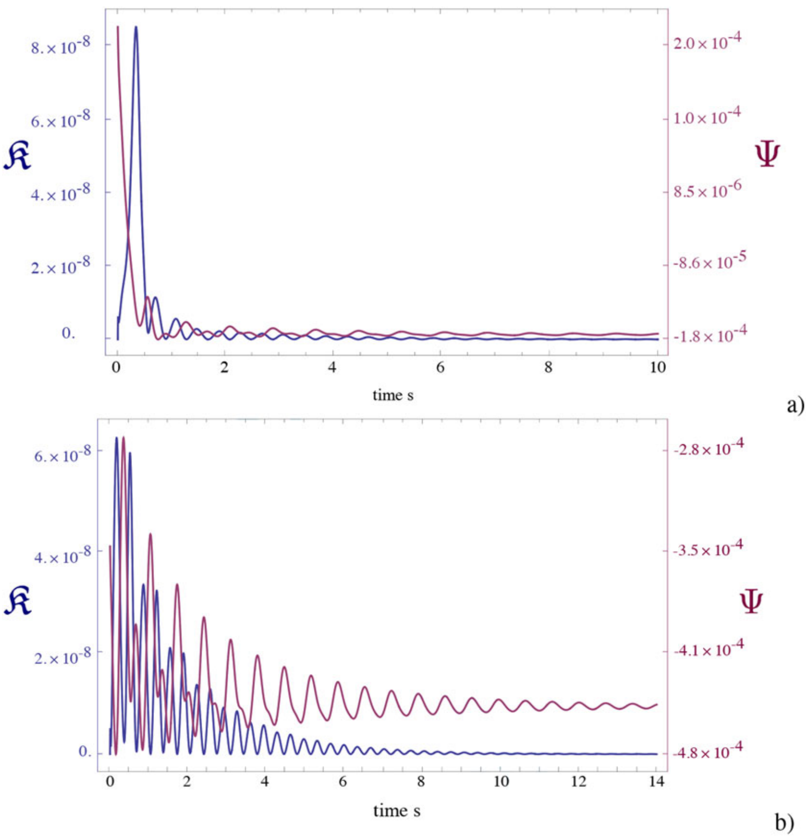

FIGURE 12 Kinetic $\mathfrak{K}$ (blue solid line) and potential energy Ψ (purple solid line) vs time graphs for the paper case **a**, and for the PET case **b**

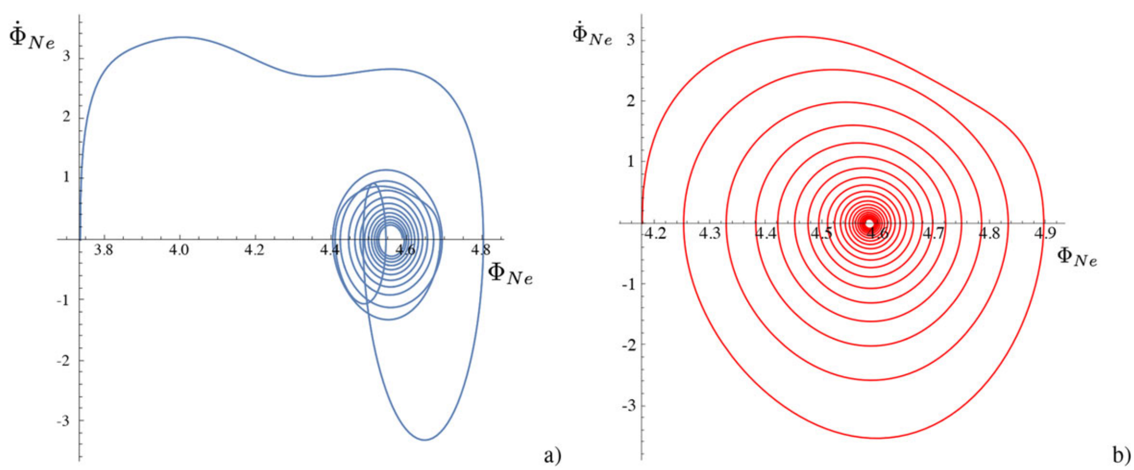

FIGURE 13 $\dot{\Phi}_{Ne}$ vs Φ_{Ne} phase paths of the paper case **a**, and of the PET case **b**

of beam elements under the effect of distributed loads (may these loads be uniform or not uniform) have not been extensively studied; ii) very few numerical and theoretical results are available; iii) to our knowledge no experimental evidence of some nonstandard (but stable) equilibrium configurations for said beams have been described. On the other hand, the dynamics of pantographic metamaterials (see [42,85]) seems to promise very interesting developments and therefore the dynamical phenomena occurring to their elementary constituents (*i.e.* beams with distributed loads) seems necessary. The main result of the present paper consists in showing experimentally that both the local minima for the total energy of Euler–Bernoulli inextensible highly-flexible beams which have been predicted in [36] correspond to real, effectively observed, equilibrium configurations for some physical objects (PET or paper strips). While the global minimum corresponds to a configuration which is expected and rather standard (we could call it the configuration ‘like the pending strip’, the other theoretically observed local minimum corresponds to a (nonstandard) curled shape. In the present paper, we show with a very simple experimental apparatus that the curled equilibrium shapes can be observed experimentally. Moreover, we show that, by means of a simple Lagrangian Hencky-type directly discrete model, it is rather easy to predict numerically both the observed equilibrium shapes and the damped oscillations (non-linear and linear) in the neighborhood of considered configurations. Remark that we wanted to use directly a Hencky-type discrete model for the prediction of the dynamic behavior of used strips as we wanted to avoid any discretization of a continuous Euler-type model. The validity of this choice is supported, also, by the very first study on the subject which one can find in [37]. In any case the adopted Lagrangian formulation i) is computationally very efficient and ii) can be interpreted from a physical point of view in a very clear way (see the previous figures describing Hencky type-models). Moreover, Lagrangian discrete models are producing very effective systems of ODEs which can be easily used to predict the motion of considered systems in the most general nonlinear regime. Besides, exactly by exploiting the said efficient features of Lagrangian discrete models, we could compare the numerically obtained solutions of the Lagrangian differential equations with the results of our experimental measurements. We have got a very good qualitative agreement (the reader can refer to the supplementary data which we have made available online: they include the videos of experimental observations and the animations obtained with our numerical codes. These videos are related to the examples in Figures 10 and 11). However, for the limited set of measurements which we could get with the required precision, we also obtained a very close correspondence between the theoretical predictions and the observed physical quantities. The reader has to remark also that the quantitative and qualitative agreement which has been obtained is based on the identification of very few constitutive parameters and that the Hencky-type discrete model has to be compared with the discretization of Euler continuous beam obtained by introducing few finite elements or isogeometric schematizations involving few degrees of freedom.

It can be therefore concluded that even when more accurate experimental set-ups will be constructed and used the modeling procedure presented here has great chances to be carefully predictive.

The Hencky-type model is based on the choice of some lumped constitutive parameters, to be introduced in the postulated Lagrangian functions: we expect that even in presence of more sophisticated dynamic measurements these parameters can be efficiently fitted to get further precise and robust quantitative predictions (in this context see, *e.g.*, [1,88,97]).

We are sure of the greater efficiency of Hencky-type models also when more complicated systems are considered (see [43]). Therefore we are confident that codes similar to the simple ones which were used in the present paper can be developed, and will be extremely useful, also in the dynamics study of i) Timoshenko beams,[30,31,60] ii) lattice systems including many beam structural elements both in the case of 1D systems,[7,67,90,113] in small or large deformations regimes,[19] or in the case 2D and 3D structures,[20,52,65,86,93,101,110] iii) more complex micro-structures which arise in the theory of metamaterials (see [53]). We believe that the alternative approach, based on the formulation of continuous models and on their subsequent discretization[18,23] although is substantially equivalent, may present some difficulties, when the discretization process of the continuous model is obtained without taking into account the physical nature of the modeled mechanical systems.

The theoretical analysis presented here allows for the determination of the sufficient criteria for which the curled equilibrium configuration can be experimentally observable (*i.e.* be stable). In this respect, one can note that the case of PET beam required a more sophisticated analysis. To get the stability of the curled configuration it was necessary to add a concentrated mass, *i.e.* the paper clip, at the free end of the strip. The distributed weight of PET strip was not enough and additional weight (we choose to place it at the free end) was required for making stable the 'curled equilibrium'. Another possibility would have been to use a longer strip, as 'by adding length' the nondimensional analysis of used Lagrange equations we could have achieved a stable equilibrium configuration also under its distributed own weight alone.

This analysis is confirmed by the observation that, after having removed the additional tip mass there is only one minimum of total energy and that the curled equilibrium configuration becomes unstable. In this case, there is not a second local minimum for total energy.

These results are confirmed by the analysis developed in [36] where it was shown that there exists a critical value for the applied external load under which only one minimum of the total energy exists.

This minimum is obviously related to the standard equilibrium configuration which is always observed (and often described in the literature) which is very similar to the 'pending wire shape' exhibited also by a flexible cantilever beam. To confirm further the aforementioned conclusion in Figure 14 we show, for the constitutive parameters characterizing the experimentally used PET strip, that, in absence of the tip mass, the subsequent configurations as calculated in a stroboscopic motion. This motion is sampled at the rate of 10 images per second starting from an imposed initial configuration. This configuration is represented by a green solid line and is the same which was displayed in Figure 9b. The minimum of the total energy represented the image of the current configuration captured at 1.4 s (red solid line). The interested reader will find a movie for this motion based on performed numerical simulation added as supplementary data in the website of the journal.

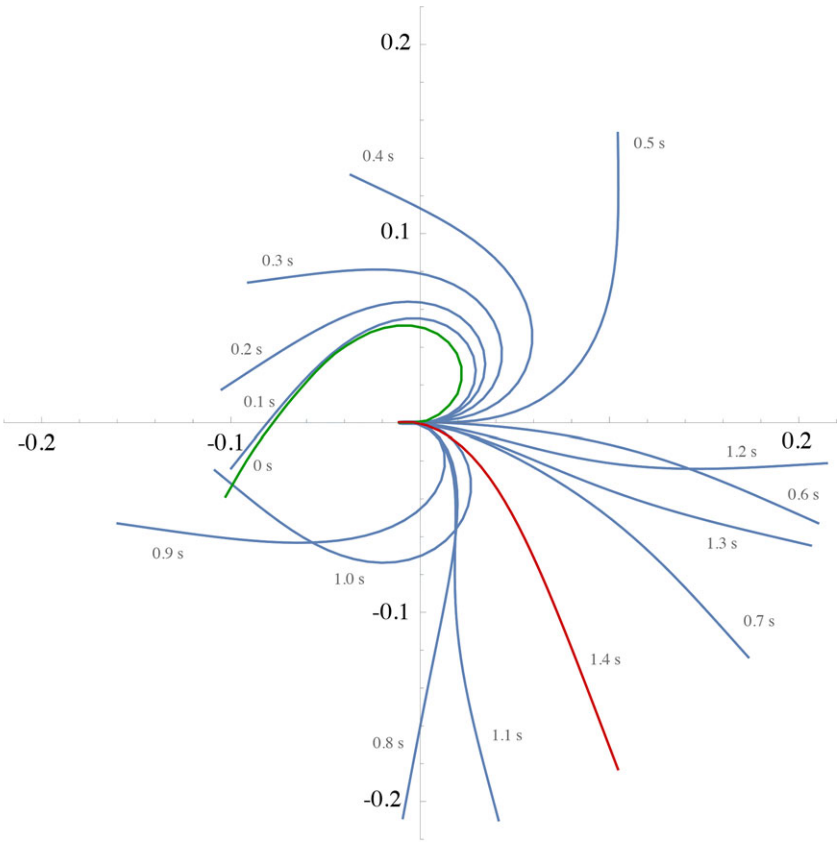

FIGURE 14 Stroboscopic motion of the PET beam without mass on the tip. The initial shape is highlighted in green and the final one in red. The label near each configuration represents the attained time for the shown configuration

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