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ORIGINAL ARTICLE

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Nonlinear travelling strain waves in a gradient-elastic medium

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Abstract In this paper, we investigate the influence of geometric nonlinearity on the propagation of longitudinal and shear waves in a gradient-elastic medium. It is shown that taking into account surface energy we observe the destruction of travelling plane longitudinal and shear waves.

Keywords Gradient-elastic medium · Surface energy · Geometric nonlinearity · Travelling wave

1 Introduction

Along with the model of the classical continuum, the models of generalized continua are widely used in the mechanics of a deformable solids, see [1–11] and the references therein. An important class of the generalized continua is a gradient-elastic medium. The appearance of this model dates back to the beginning of the twentieth century and is associated with the names of Le Roux [12,13] and Jaramillo [14]. It is worth to note that the famous Cosserat continuum model [15] with constraint rotations, that is when the rotation vector depends on the displacement curl [16], can be also treated as a gradient-elastic medium. The most general model of a gradient-elastic medium is considered by Mindlin [17]. However, its practical use is hampered by the introduction in this theory of a large number of elastic moduli that require experimental determination. In the works by Vardoulakis and his colleagues [18–21], a modified version of Mindlin's theory was proposed, which leads to significant simplification of this mathematical model while its basic properties are retaining.

Some gradient models allow us to take into account the surface energy [22–30], the same relates the models proposed in [18–21], i.e., the excess energy of the surface layer at the surface/interface, due to the difference in intermolecular interaction in both phases. If the surface is free, then the role of the second phase is played by a vacuum.

Mathematical models of structured media with surface energy may be useful in studying the acoustic properties of materials and structural elements in which new surface areas have been formed during the development of cracks or the surface has contracted during adhesion of various bodies. These processes lead to a change in the mutual arrangement of atoms, which leads to a change in the surface energy and the stress-strain state.

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2 Mathematical model

For linear elastic materials, the Mindlin theory [17] provides a general basis for developing gradient theories of deformations. Therefore, in this section, we first briefly recall the basic equations of this theory and then give them a modified version according to the approach developed by Vardoulakis and his colleagues [18–21]. The idea of an elementary cell (microenvironment) is introduced into the theory by Mindlin, which can be interpreted as a periodic structure of the crystal lattice, polymer molecules, polycrystal crystallites or grains of granular material. Then, the corresponding kinematic variables are determined to describe geometric changes in both macro- and microenvironment. Then, relative to the Cartesian coordinate system $Ox_1x_2x_3$, the following dependence is taken for the density of potential energy (potential energy per unit of macro-volume):

$$W = W\left(\varepsilon_{qr}, \gamma_{qr}, \chi_{qrs}\right) \tag{1}$$

where $\varepsilon_{qr} = \partial_r U_q + \partial_q U_r$ is the strain tensor, U_q are the components of displacement vector, $\partial_r \equiv \partial/\partial x_r$, the indices (q, r, s) are in the range (1; 2; 3), $\gamma_{qr} = \partial_q U_r - \psi_{qr}$ is the relative strain with ψ_{qr} denoting the micro deformation (i.e., the displacement-gradient in the micro-medium), and $\chi_{qrs} \equiv \partial_q \psi_{rs}$ is the micro-deformation gradient. Then, appropriate definitions for the stresses follow from the analysis of the first variation of W:

$$\tau_{qr} \equiv \frac{\partial W}{\partial \varepsilon_{qr}}, \quad \alpha_{qr} \equiv \frac{\partial W}{\partial \gamma_{qr}}, \quad \mu_{qrs} \equiv \frac{\partial W}{\partial \chi_{qrs}} \tag{2}$$

where $\left(\tau_{qr}, \alpha_{qr}, \mu_{qrs}\right)$ are the Cauchy stress (symmetric), relative stress (asymmetric), and double stress tensors, respectively.

In what follows, from the variational equation of motion considering independent variations δU_q and $\delta\psi_{qr}$ and assuming that the micro-medium cell is a cube with edges of length $2h$, one may obtain the following twelve stress scalar equations of motion [17]:

$$\partial_q \sigma_{qr} + f_r = \rho'(\partial_{tt} U_r), \tag{3}$$

$$\partial_q \mu_{qrs} + \alpha_{rs} + \Phi_{rs} = \frac{1}{3}\rho h^2 \left(\partial_{tt}\psi_{rs}\right), \tag{4}$$

and the twelve traction boundary conditions

$$t_r = n_q \sigma_{qr}, \quad T_{rs} = n_q \mu_{qrs}, \tag{5}$$

where $\rho' \equiv \rho_M + \rho,\ \rho_M$, is the mass of macro-material per unit macro-volume, ρ is the mass of micro-material per unit macro-volume, $\sigma_{qr} \equiv \tau_{qr} + \alpha_{qr}$ is the total stress tensor, n_q are the components of the unit vector of outer normal to the boundary, f_r is the body force per unit volume and t_r is the surface force per unit area, Φ_{rs} is the double force per unit volume, see e.g., Love [31] for an interpretation of this force system, and T_{rs} is the double force per unit area, and ∂_t denotes time differentiation.

However, we should mention that the above formulation involves, in its general form, a very large number of elastic moduli and, therefore, applying it to practical situations may be extremely difficult, see for example, [27]. The particular form proposed in [18–21] can be considered as one of the simplest versions of Mindlin's elasticity theory with microstructure.

More specifically, Vardoulakis and co-workers suggested the following form for the strain-energy density function [18–21]:

$$W = \frac{1}{2}\lambda\varepsilon_{qq}\varepsilon_{rr} + \mu\varepsilon_{qr}\varepsilon_{rq} + \mu c\left(\partial_m\varepsilon_{qr}\right)\left(\partial_m\varepsilon_{rq}\right) + \mu b_m \partial_m\left(\varepsilon_{qr}\varepsilon_{rq}\right), \tag{6}$$

where λ and μ are classic Lame moduli, c, b are additional elastic moduli which characterize a gradient medium, $b_m = b\vartheta_m, \vartheta_m\vartheta_m = 1, \partial_m$ means differentiation with respect to the coordinate x_m, ε_{qr} are components of the strain tensor, indices q, r, m are taken from the range $\overline{1, 3}$.

The last term on the right-hand side of (6) refers to the surface energy, since by the Gauss–Ostrogradsky theorem it can be written in the form

$$\int_{\Omega} \partial_m \left(b_m \varepsilon_{qr} \varepsilon_{rq}\right) \mathrm{d}\Omega = b \int_{S} \left(\varepsilon_{qr} \varepsilon_{rq}\right) \left(\vartheta_m n_m\right) \mathrm{d}S,$$

where S is the boundary of the volume Ω.

The positive definiteness of (1) requires the following restrictions on material moduli:

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad c > 0, \quad -1 < b/c^{1/2} < 1.$$

Modulus c depends on a size on the structural elements and equals to $c = (h/4)^2$, whereas modulus b can be defined as the material length-scale parameter related to surface energy.

With the help of (2) and (6), we can express stresses and double stresses through the components of the strain tensor as follows:

$$\begin{aligned} \tau_{qr} &= \lambda \delta_{qr} \varepsilon_{mm} + 2\mu \varepsilon_{qr} + 2\mu b_m \left(\partial_m \varepsilon_{qr}\right), \\ \mu_{mqr} &= 2\mu \left[b_m \varepsilon_{qr} + c\varepsilon_{qr,m} + \left(b_m \varepsilon_{qn} + c\varepsilon_{qn,m}\right) U_{r,n}\right]. \end{aligned} \tag{7}$$

Within the variational approach, considering $\boldsymbol{\delta U_q}$ as an independent variation we obtain the equations of motion and natural boundary conditions in the stresses in the case of a regular boundary:

$$\begin{aligned} \partial_q \sigma_{qr} &= \rho \frac{\partial^2 U_r}{\partial t^2}, \\ \partial_q \mu_{mqr} + \alpha_{rm} &= I \frac{\partial^2 \psi_{rm}}{\partial t^2}, \\ n_r \tau_{rm} - n_q n_r n_m \partial_m \mu_{qrm} - 2n_r \left(\delta_{ql} - n_q n_l\right) \partial_l \mu_{qrm} &+ \left(n_q n_r n_l \left(\delta_{lq} - n_l n_j\right) \partial_j - n_q \left(\delta_{rl} - n_l l\right) l\right) \mu_{qrm} \\ &+ \frac{1}{3} \rho h^2 n_r \frac{\partial^2 \psi_{rm}}{\partial t^2} = P_m, \\ n_q n_r \mu_{qrm} &= R_m, \end{aligned} \tag{8}$$

where $\boldsymbol{\delta_{qr}}$ is the Kronecker symbol, $\boldsymbol{P_m}$ is a surface force vector per unit area, $\boldsymbol{R_m}$ is a double surface force without moment per unit area.

If we take into account the geometric nonlinearity in (1), i.e., if we introduce into consideration the tensor of finite deformations $\boldsymbol{\varepsilon_{qr} = \partial_r U_q + \partial_q U_r + \partial_r U_m \partial_q U_m}$, the form of Eq. (8) will not change.

Taking into account relations (7) and (8), we obtain the equation of motion in displacements:

$$\rho \frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \frac{1}{3} \rho h^2 \Delta \frac{\partial^2 \boldsymbol{U}}{\partial t^2} + (\lambda + \mu - \mu c \Delta) \operatorname{grad} \operatorname{div} \boldsymbol{U} + \mu \Delta \boldsymbol{U} - \mu c \Delta^2 \boldsymbol{U} + \boldsymbol{f}. \tag{9}$$

The vector $\boldsymbol{f}$ contains nonlinear terms due to relation for strain tensor, the explicit form of which will be given below for those particular cases that will be considered.

3 Dispersion of longitudinal and shear waves

If in the expression for strain tensor we neglect nonlinear terms (small strain tensor), then $\boldsymbol{f} = \boldsymbol{0}$ and the dynamics of the medium corresponding to (6) will be described by the vector equation in terms of displacements:

$$\rho \frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \frac{1}{3} \rho h^2 \Delta \frac{\partial^2 \boldsymbol{U}}{\partial t^2} + (\lambda + \mu - \mu \Delta) \operatorname{grad} \operatorname{div} \boldsymbol{U} + \mu \Delta \boldsymbol{U} - \mu c \Delta^2 \boldsymbol{U}. \tag{10}$$

Let us consider plane longitudinal waves propagating in unbounded space along the direction of the $\boldsymbol{x_1}$ axis. The equation which describes these waves follows from Eq. (10) with the substitution $\boldsymbol{U} = \left(\boldsymbol{U_1}\left(\boldsymbol{x_1}, \boldsymbol{t}\right), \boldsymbol{0}, \boldsymbol{0}\right)$. As a result, we get the following equation:

$$(\lambda + \mu) \frac{\partial^2 U_1}{\partial x_1^2} - 2\mu c \frac{\partial^4 U_1}{\partial x_1^4} + \frac{1}{3} \rho h^2 \frac{\partial^4 U_1}{\partial x_1^2 \partial t^2} - \rho \frac{\partial^2 U_1}{\partial t^2} = 0. \tag{11}$$

We are looking for the solution to Eq. (11) in the form of a travelling harmonic wave:

$$U_1 = Ae^{i(kx_1-\omega t)} + c.c., \tag{12}$$

where $\boldsymbol{k}$ is a wave number, $\boldsymbol{\omega}$ is a frequency and $\boldsymbol{c.c.}$ means complex conjugated value. After substitution expression (12) into (11), we get the dispersion equation:

$$(\lambda + \mu)\,k^2 - 2\mu ck^4 + \left(Ik^2 - \rho\right)\omega^2 = 0, \tag{13}$$

from which we find an explicit dependence of the frequency on the wave number:

$$\omega = k\sqrt{\frac{c_l^2 + 2cc_\tau^2 k^2}{1 + h^2k^2/3}}, \quad c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_\tau = \sqrt{\frac{\mu}{\rho}}, \tag{14}$$

where $\boldsymbol{c_l}$ and $\boldsymbol{c_\tau}$ are velocities of longitudinal and shear waves in the absence of a microstructure.

Let us denote the phase velocity as $\boldsymbol{C}$ with upper index stand for the type of a wave ($\boldsymbol{l}$ is for longitudinal one, $\boldsymbol{\tau}$ is for shear wave). We introduce the dimensionless frequency, wave number and phase velocity as follows

$$k_d = k\sqrt{c}, \omega_d = \omega\frac{h}{\sqrt{3}c_\tau}, C_d = \frac{k_d}{\omega_d} \tag{15}$$

which we further use in the study of various types of waves.

In dimensionless values, the phase velocity of the longitudinal wave is given by:

$$C_d^l = \sqrt{\frac{(c_l/c_\tau)^2 + 2k_d^2}{3/16 + k_d^2}}.$$

At small values of the wave number, i.e., when the size of the microelement does not affect the wave process, there is no dispersion. In this case, the phase velocity $\boldsymbol{C^l}$ coincides with the velocity of a longitudinal wave in a classical elastic medium. When $\boldsymbol{\omega} \to \infty$ dispersion is also absent and the asymptotic value of the phase velocity of the longitudinal wave is given by

$$C^l = \frac{\sqrt{3}}{2\sqrt{2}}c_\tau, \quad C_d^l = \sqrt{2}.$$

It is worth to note that the Cosserat medium model does not describe the dispersion of a longitudinal wave at all, and the Le Roux medium model, which describes a dispersion, leads to the infinite growth of phase velocity at $\boldsymbol{\omega} \to \infty$ [16].

The equation describing the propagation of a plane shear wave can be obtained from (11) with the substitution $\boldsymbol{U} = (\boldsymbol{0}, \boldsymbol{U_2}(\boldsymbol{x_1}, \boldsymbol{t}), \boldsymbol{0})$. It reads

$$\mu\frac{\partial^2 U_2}{\partial x_1^2} - \mu c\frac{\partial^4 U_2}{\partial x_1^4} + \frac{1}{3}\rho h^2\frac{\partial^4 U_2}{\partial x_1^2\partial t^2} - \rho\frac{\partial^2 U_2}{\partial t^2} = 0. \tag{16}$$

Similarly, as for the longitudinal wave, we find the dispersion relation which takes the form

$$\omega = kc_\tau\sqrt{\frac{1 + ck^2}{1 + h^2k^2/3}}. \tag{17}$$

The phase velocity of the shear wave, dimensionless with the help of (12), is given now as follows:

$$C_d^l = \sqrt{\frac{(c_l/c_\tau)^2 + 2k_d^2}{3/16 + k_d^2}}.$$

The coincidence of the dispersion relations, built according to the above formulas and the classical theory of elasticity, is observed for small values of $\boldsymbol{k}$ and $\boldsymbol{\omega}$. In this case, the dispersion is absent and the phase velocity

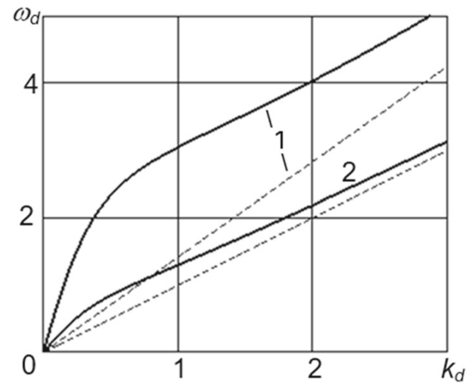

Fig. 1 Dispersion relations for the longitudinal (1) and shear (2) waves propagating in a gradient-elastic medium

$\boldsymbol{C^\tau} = \boldsymbol{c_\tau}$. With the increase in frequency the phase velocity decreases and at the limit $\boldsymbol{\omega} \to \infty$, it takes the following value:

$$C^\tau = \frac{\sqrt{3}}{4} c_\tau, \quad C_d^\tau = 1.$$

In Fig. 1 for the medium with parameter $\boldsymbol{r} = \frac{\boldsymbol{\lambda}}{\boldsymbol{\mu}} = \mathbf{3}$ are shown the dependences of the normalized frequency on the normalized wave number of the longitudinal (curve 1) and shear (curve 2) waves and their asymptotes (dashed straight lines).

The dispersion equation for a longitudinal wave in dimensionless quantities contains only one parameter $\boldsymbol{r}$, which is necessary for solving this equation. The form of the dispersion curve for different values of $\boldsymbol{r}$ does not change; however, with the increase in the parameter $\boldsymbol{r}$, the ratio of the phase velocities of the longitudinal and transverse waves increases.

Note that in (10) there are no terms with the parameter $\boldsymbol{b}$. The velocities of longitudinal and shear waves also do not depend on this parameter, i.e., the additional term in the potential energy density expression, which is responsible for the surface energy, does not affect the propagation of linear bulk waves in the studied model of the medium.

4 Nonlinear longitudinal wave

In the nonlinear formulation (2) and (6), we consider the problem of the propagation of plane longitudinal and shear waves. First, longitudinal wave motions are investigated, in which the particles of the medium move in the direction of motion. Let us use the direction of motion the axis x_1. The displacement vector will have the form $\boldsymbol{U} = (U_1(x_1, t), 0, 0)$.

Nonlinear longitudinal motions can be described with the help of the following equation:

$$(\lambda + 2\mu) \frac{\partial^2 U_1}{\partial x_1^2} - 2\mu c \frac{\partial^4 U_1}{\partial x_1^4} + \frac{1}{3}\rho h^2 \frac{\partial^4 U_1}{\partial x_1^2 \partial t^2} - \rho \frac{\partial^2 U_1}{\partial t^2}$$
$$= -\frac{\lambda + 2\mu}{2} \frac{\partial}{\partial x_1} \left(\frac{\partial U_1}{\partial x_1} \right)^2 + 2\mu c \frac{\partial^3}{\partial x_1^3} \left(\frac{\partial U_1}{\partial x_1} \right)^2 + 2\mu b_1 \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial U_1}{\partial x_1} \right)^2. \quad (18)$$

The propagation of plane longitudinal waves described by this equation will be affected by two factors: dispersion and nonlinearity. Nonlinearity leads to the emergence in the wave of new harmonics, into which the energy from the main disturbance is continuously pumped. This contributes to the appearance of sudden changes in the propagating wave profile. On the contrary, the dispersion makes the profile smoother due to the difference in the phase velocities of the harmonic components of the wave. The combined effect of these two factors, their so-called "competition," can lead to the formation of travelling waves. Such waves propagate at a constant speed without changing of their shape.

We seek solutions to this equation in the class of stationary deformation waves $\varepsilon(\xi) = \frac{\partial U_1}{\partial \xi}$, where $\xi = x - Vt$ is the travelling coordinate, $V = const$ is velocity of travelling wave, which is a priori unknown.

Then (5) reduces to a nonlinear ordinary differential equation that is insoluble relative to the highest derivative

$$\frac{h^2}{3}\left(V^2-\frac{3}{8}C_\tau^2\right)\frac{d^2\varepsilon}{\partial\xi^2}+\left(C_l^2-V^2\right)\varepsilon=-C_l^2\varepsilon^2+2C_\tau^2 b_1\frac{d\left(\varepsilon^2\right)}{\mathrm{d}\xi}+\frac{C_\tau^2h^2}{8}\frac{d^2\left(\varepsilon^2\right)}{\mathrm{d}\xi^2}, \tag{19}$$

Let us carry out a qualitative analysis of the behaviour of solutions of this equation on the phase plane $\left(\varepsilon,\frac{\mathrm{d}\varepsilon}{\mathrm{d}\xi}\right)$. We assume at the first stage that the surface energy is negligible, i.e., the parameter responsible for the surface energy is zero. Depending on the ratio of the speed of a nonlinear wave V and velocities of longitudinal and shear waves C_l and C_τ the behaviour of solutions will be qualitatively different. The phase portraits of Eq. (16) are shown in Fig. 2.

The approaches for constructing and analysing phase portraits of nonlinear dynamic systems described by ordinary second-order differential equations can be found, for example, in [32].

When $V<\sqrt{3/8}C_\tau$ closed phase trajectories are absent; there are no bounded solutions of (19) (Fig. 2a).

In case when $\sqrt{3/8}C_\tau<V<C_l$, there are closed phase trajectories (Fig. 2b). At the origin, there is a singular point of the "centre" type, to the left, where there is a singular point of the "saddle" type. Straight line

$$\varepsilon=\frac{4}{3}\frac{V^2}{C_\tau^2}-\frac{1}{2}, \tag{20}$$

relates to the region of forbidden motions along the phase trajectories. Equation (16) has bounded periodic solutions. Thus, in this case there exist periodic nonlinear travelling waves.

When $V>C_l$ there are closed phase trajectories (Fig. 2c). At the origin, there is a singular point of the "saddle" type, to the right of which there is a singular point of the "centre" type. The straight line given by relation (20) is still is the boundary of the region of forbidden motions. In this case, it is possible to have a solitary nonlinear travelling wave, which is a soliton of deformation of positive polarity. Soliton of amplitude A, of width Δ and of velocity V which relate by the following formulae:

$$A\sim\frac{3h^2\left(8C_l^2-3C_\tau^2\right)}{C_l^2\left(8h^2-6\Delta^2\right)},\quad \Delta^2\sim\frac{4h^2\left(V^2-3/8C_\tau^2\right)}{3\left(V^2-C_l^2\right)}. \tag{21}$$

From (21), we establish that the soliton amplitude is determined by its velocity for fixed λ, μ, ρ. In its turn, the velocity of a solitary wave depends on Δ and h. Figure 3a shows the dependence of the amplitude on the width of the soliton at a fixed cell size. The dependence of the width of the localized motion on its velocity and cell size is shown in Fig. 3b.

The behaviour of a soliton is classical, since a wave of greater amplitude has a smaller width and propagates with higher velocity. Minimum value $\Delta=2h/\sqrt{3}$ is achieved with an infinite growth of the soliton velocity.

From the last relation in (21), we can note a linear dependence of the soliton width on h. When the velocity of a solitary wave tends to the velocity of a longitudinal wave without taking into account the microstructure, the soliton amplitude tends to zero, whereas its width tends to infinity.

The account of terms with coefficient b_1 introduces non-conservatism into the system. Special points of the "centre" type are transformed into "focuses," and, the larger the coefficient responsible for the surface energy, the faster the phase trajectories fall into a singular point. Separatrices of "saddles" also change, and this leads to the destruction of solitons. We can say that if the coefficient $b_1=0$ and the surface energy does not affect the medium, then travelling strain waves are possible. Otherwise, they are not possible.

5 Nonlinear shear wave

Finally, let us consider the effect of geometric nonlinearity on transverse motion. The displacement vector for vertically polarized motions has the form $\boldsymbol{U}=(0,U_2(x_1,t),0)$. Nonlinear plane transverse motions will be described by the following equation:

$$\mu\frac{\partial^2U_2}{\partial x_1^2}-\mu c\frac{\partial^4U_2}{\partial x_1^4}+\frac{1}{3}\rho h^2\frac{\partial^4U_2}{\partial x_1^2\partial t^2}-\rho\frac{\partial^2U_2}{\partial t^2}=\frac{2\mu c}{3}\frac{\partial^3}{\partial x_1^3}\left(\frac{\partial U_1}{\partial x_1}\right)^3+\mu b_1\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial U_1}{\partial x_1}\right)^3, \tag{22}$$

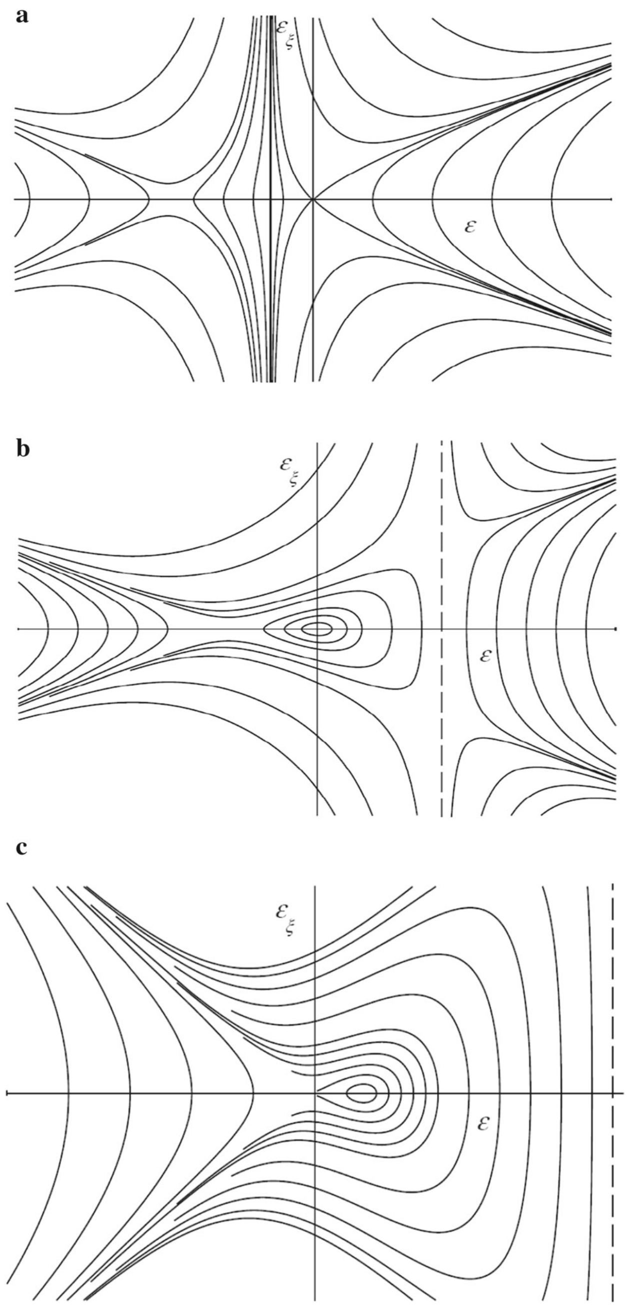

Fig. 2 **a** Phase portrait $\left(\varepsilon, \frac{\mathrm{d}\varepsilon}{\mathrm{d}\xi}\right)$, $V < \sqrt{3/8}C_\tau$. **b** Phase portrait $\left(\varepsilon, \frac{\mathrm{d}\varepsilon}{\mathrm{d}\xi}\right)$, $\sqrt{3/8}C_\tau < V < C_l$ $(V = C_\tau)$. **c** Phase portrait $\left(\varepsilon, \frac{\mathrm{d}\varepsilon}{\mathrm{d}\xi}\right)$, $V > C_l$ $(V = 2C_\tau)$

Travelling shear waves $E(\xi) = \frac{\partial U_2}{\partial \xi}$ will be described by an ordinary differential equation that cannot be solved with respect to the highest derivative:

$$h^2 \left(\frac{V^2}{3C_\tau^2} - \frac{1}{16}\right) \frac{d^2 E}{\partial \xi^2} + \left(1 - \frac{V^2}{C_\tau^2}\right) E = b_1 \frac{d\left(E^2\right)}{\mathrm{d}\xi} + \frac{h^2}{24} \frac{d^2\left(E^3\right)}{\mathrm{d}\xi^2}. \tag{23}$$

Depending on the relationship V/C_τ, the behaviour of the solutions of this equation on the phase plane $\left(E, \frac{\mathrm{d}E}{\mathrm{d}\xi}\right)$, will be qualitatively different. Bounded solutions (when $b_1 = 0$) are possible only in the case when $\sqrt{3}C_\tau/4 < V < C_\tau$. The corresponding phase portrait is shown in Fig. 4.

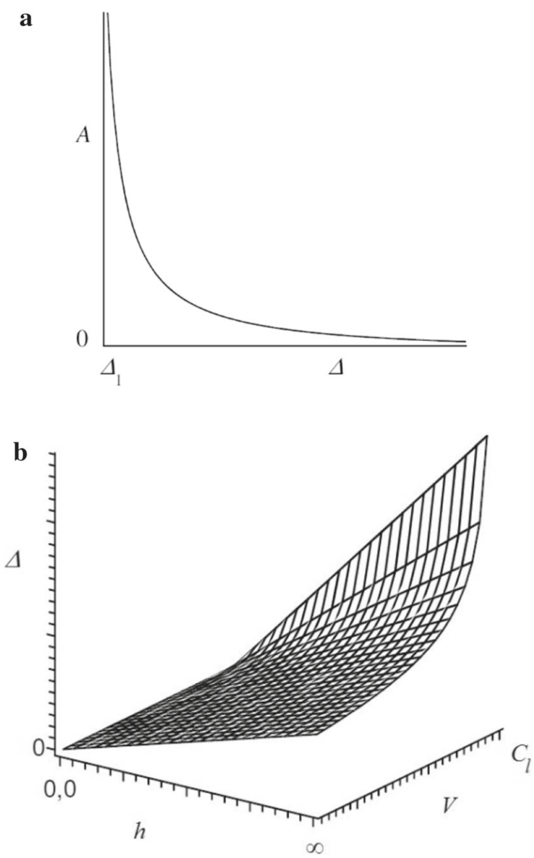

Fig. 3 **a** Dependence $A\,(\Delta)$, $\Delta_1 = 2h/\sqrt{3}$. **b** Dependence $\Delta\,(V, h)$

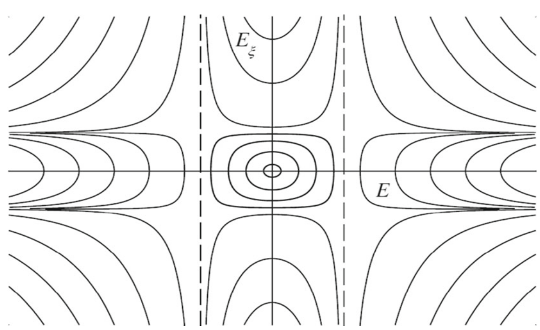

Fig. 4 Phase portrait $\left(E, \frac{dE}{d\xi}\right)$, $\sqrt{3}C_\tau/4 < V < C_\tau$

Straight lines $E = \pm\sqrt{\frac{1}{2}\left(\frac{16}{3}\frac{V^2}{C_\tau^2} - 1\right)}$ represent the boundaries of areas of forbidden motions on the phase plane. Travelling waves of shear deformation can be only periodic. There are no solitons of shear deformations.

The presence of b_1 leads to the transformation of a singular point of the "centre" type into a singular point of the "focus" type, which leads to the destruction of the wave. So in a gradient-elastic medium with a surface energy in the case $b_1 \neq 0$, there are no stationary waves.

6 Conclusions

During last two decades, many facts of nonlinear behaviour of rocks have been experimentally discovered [31,32] using seismic techniques. Among such facts there are wave front reversal [33], seismic emission [31], amplitude-dependent attenuation [34], a decrease in the wave propagation velocity and an increase in their amplitude (the so-called "bright spot" phenomenon) [35].

To explain the above and other observed phenomena, linear elasticity-based mathematical are not enough models for proper description of wave processes in rocks. At the same time, mathematical models require generalization both for the case of taking into account nonlinearities (geometric, physical, cavity, contact-type) and for taking into account the dissipative–dispersive properties of materials.

In a number of works [18,27,36–38], the efficiency of the models of gradient elasticity for describing dynamic processes in soils and rocks has been shown. In the present work, a nonlinear gradient-elastic model of a solid body was used to describe intense longitudinal and shear waves of a stationary profile. The influence of surface energy on the stability of such waves was discussed. It is shown that the propagation of longitudinal and shear waves described by the equations of the gradient theory of elasticity will be affected by two factors: dispersion and nonlinearity. When they interact, travelling waves can be formed that propagate at a constant speed and do not change their profile. Accounting for surface energy leads to the destruction of travelling waves.

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References

1. Maugin, G.A., Metrikine, A.V. (eds.): Mechanics of Generalized Continua: On Hundred Years After the Cosserats. Advances in Mathematics and Mechanics, vol. 21, p. 338. Springer, Berlin (2010)
2. Altenbach, H., Maugin, G.A., Erofeev, V. (eds.): Mechanics of Generalized Continua. Advanced Structured Matherials, vol. 7, p. 350. Springer, Berlin (2011)
3. Altenbach, H., Forest, S., Krivtsov, A. (eds.): Generalized Continua as Models with Multi-scale Effects or Under Multi-field Actions. Advanced Structured Matherials, vol. 22, p. 332. Springer, Berlin (2013)
4. Altenbach, H., Eremeyev, V.A. (eds.): Generalized Continua—From the Theory to Engineering Applications, p. 388. Springer, Wien (2013)
5. Bagdoev, A.G., Erofeyev, V.I., Shekoyan, A.V.: Wave Dynamics of Generalized Continua. Advanced Structured Matherials, vol. 24, p. 274. Springer, Berlin (2016)
6. Altenbach, H., Forest, S. (eds.): Generalized Continua as Models for Classical and Advanced Materials. Advanced Structured Matherials, vol. 42, p. 458. Springer, Zürich (2016)
7. Maugin, G.A.: Non-Classical Continuum Mechanics. Advanced Structured Matherials, vol. 51, p. 260. Springer, Singapore (2017)
8. dell'Isola, F., Eremeyev, V.A., Porubov, A. (eds.): Advanced in Mechanics of Microstructured Media and Structures. Advanced Structured Matherials, vol. 87, p. 370. Springer, Cham (2018)
9. Altenbach, H., Pouget, J., Rousseau, M., Collet, B., Michelitsch, T. (eds.): Generalized Models and Non-Classical Approaches in Complex Materials 1. Advanced Structured Matherials, vol. 89, p. 760. Springer, Cham (2018)
10. Altenbach, H., Pouget, J., Rousseau, M., Collet, B., Michelitsch, T. (eds.): Generalized Models and Non-Classical Approaches in Complex Materials 2. Advanced Structured Matherials, vol. 90, p. 306. Springer, Cham (2018)
11. Erofeev, V., Porubov, A., Sargsyan, S. (Editors). Nonlinear Wave Dynamics of Generalized Continua. Materials Physics and Mechanics. (2018). Vol. 35, No 1 (Spesial Issue dedicated to the memory E.L. Aero and G. Maugin). 190 p
12. Le Roux, J.: Etude geometrique de la flexion, dans les deformations infinitesimaleg d'nn milien continu. Annales de l'Ecole Normale Supereure. 3e serie. (1911). Tome 28. P.523-579
13. Le Roux, J.: Recherchesg sur la geometrie beg deformatios finies. Annales de l'Ecole Normale Supereure. 3e serie. Tome 30, pp. 193–245 (1913)
14. Jaramillo, T.J.: A generalization of the energy function of elasticity theory. Dissertation, Department of Mathematics, University of Chicago, p. 154 (1929)
15. Cosserat, E., et al.: Theorie des Corp Deformables, p. 226. Librairie Scientifique A. Hermann et Fils, Paris (1909)
16. Erofeyev, V.I.: Wave Processes in Solid with Microstructure. Stability, Vibration and Control of Systems, vol. 8, p. 256. World Scientific, New Jersey (2003)
17. Mindlin, R.D.: Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. **16**, 51–78 (1964)
18. Vardoulakis, I., Sulem, J.: Bifurcation Analysis in Geomechanics. Blackie Academic and Professional, London (1995)
19. Vardoulakis, I., Exadaktylos, G., Aifantis, E.: Gradient elasticity with surface energy: mode III crack problem. Int. J. Solids Struct. **33**, 4531–4559 (1996)
20. Exadaktylos, G., Vardoulakis, I., Aifantis, E.: Cracks in gradient elastic bodies with surfaceenergy. Proc. 4th Natl. Greek Conference on Mechanics. (eds. P.S. Theocaris and E.E. Gdoutos) **1**, 341–351 (1995)
21. Vardoulakis, I., Georgiadis, H.G.: SH surface waves in a homogeneous gradient-elastic half-space with surface energy. J. Elast. **47**, 147–165 (1997)
22. Georgiadis, H.G., Velgaki, E.G.: High-frequency Rayleigh waves in materials with micro-structure and couple-stress effects. Int. J. Solids Struct. **40**, 2501–2520 (2003)
23. Belov, P.A., Lurie, S.A.: Theory of ideal adhesive interactions. J. Compos. Mech. Des. **13**(4), 519–534 (2007)

24. Lurie, S., Zubov, V., Tuchkova, N., Volkov-Bogorodsky, D.: Advanced theoretical and numerical multiscale modeling of cohesion/adhesion interactions in continuum mechanics and its applications for filled. Comput. Mater. Sci. **45**(3), 709–714 (2009)
25. Frolenkova, L.Y., Shorkin, V.S.: Surface energy and adhesion energy of elastic bodies. Mech. Solids **52**(1), 62–74 (2017)
26. Eremeyev, V.A., Rosi, G., Naili, S.: Surface/interfacial anti-plane waves in solids with surface energy. Mech. Res. Commun. **74**, 8–13 (2016)
27. Eremeyev, V.A., Rosi, G., Naili, S.: Comparison of anti-plane surface waves in strain-gradient materials and materials with surface stresses. Math. Mech. Solids **24**(8), 2526–2535 (2019). https://doi.org/10.1177/1081286518769960
28. Eremeyev, V.A., Sharma, B.L.: Anti-plane surface waves in media with surface structure: discrete versus continuum model. Int. J. Eng. Sci. **143**, 33–38 (2019)
29. Sharma, B.L., Eremeyev, V.A.: Wave transmission across surface interfaces in lattice structures. Int. J. Eng. Sci. **145**, 103173 (2019). https://doi.org/10.1016/j.ijengsci.2019.103173
30. Li, Y., Wei, P., Tang, Q.: Reflection and transmission of elastic waves at the interface between two gradient-elastic solids with surface energy. Eur. J. Mech. A Solid **52**, 54–71 (2015)
31. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, 4th edn. Dover Publ, New York (1944)
32. Butenin, N.V.: Elements of the Theory of Nonlinear Oscillations, p. 226. Blaisdell Publ. Com, New York (1965)
33. Kurlenya, M.V., Oparin, V.N.: Problems of nonlinear geomechanics. Part I J. Min. Sci. **35**(3), 216–230 (1999)
34. Mashinskii, E.I.: Anomalies of low-intensity acoustic wave attenuation in rocks. J. Min. Sci. **44**(4), 345–352 (2008)
35. Kurlenya, M.V., Oparin, V.N.: Problems of nonlinear geomechanics. Part II J Min. Sci. **36**(4), 305–326 (2000)
36. Nikolaevskiy, V.N.: Geomechanics and Fluidodynamics, p. 350. Kluwer Academic Publishers, Dordrecht (1996)
37. Suknev, S.V., Novopashin, M.D.: Gradient approach to rock strength estimation. J. Min. Sci. **35**(4), 381–386 (1999)
38. Preobrazhenskii, V.L.: Parametrically phase-conjugate waves: applications in nonlinear acoustic imaging and diagnostics. Physics-Uspekhi **49**(1), 98–102 (2006)